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Instabilities of Noncommutative Two Dimensional BF Model

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`Abstract` : The noncommutative extension of two dimensional BF model is considered. It is shown that the realization of the noncommutative map via the Groenewold-Moyal star product leads to instabilities of the action, hence to a non renormalizable theory.

1 Introduction

When attempting to define a noncommutative quantum field theory [1] and wishing also to arrive at a formulation which allows explicit amplitude computation, one is faced with the problem of choosing a precise form for the non commutative product. One of the most popular choices is the Groenewold-Moyal product [2, 3] which is implemented with a simple exponential formula and needs the introduction of an antisymmetric constant tensor $\theta^{\mu\nu}$ having the dimensions of an inverse mass squared. It is commonly accepted that this procedure leads to a well defined noncommutative theory if the commutative model we begin with is sound. We shall show that this is not always the case by providing a counterexample. We analyze the topological BF model in two spacetime dimensions [4, 5, 6] having in mind that a sound noncommutative extension should be based on the functional identities encoding the symmetries, on locality and power counting, just as it happens in the commutative case. This procedure, in the standard case, leads to the stability and anomaly analysis i.e. the model is perturbatively renormalizable if the classical action is the most general local functional compatible with the above constraints (stability) and the symmetries are not broken by the radiative corrections (anomaly) [7]. Since our goal is to provide a counterexample, we concentrate on the stability aspect and only at the first order in $\theta^{\mu\nu}$. Accordingly we shall not try to be exhaustive but we will explicitly show that the noncommutative extension of the two dimensional BF model based on the Groenewold-Moyal product is unstable. To fix the notation we briefly recall the functional equations (BRS identity, Landau gauge, ghost equation and vector supersymmetry) which form a closed algebraic structure and completely define, together with locality and power counting, the commutative model [6]. The first step towards a noncommutative definition is then to extend the algebraic constraints when $\theta^{\mu\nu}$ is present. Although the choice might not be unique we adopt a "minimal" extension for each functional operator and conclude that, in order to preserve the algebra, no $\theta^{\mu\nu}$ contribution is allowed. Hence the defining equations remain exactly the same we have in the commutative case. Of course this is not so for the classical action which acquires, at the first order in $\theta^{\mu\nu}$, a local contribution $(X_{\mu\nu})$ with canonical dimension equal to four and coupled to $\theta^{\mu\nu}$ itself. We then proceed to check that the usual Groenewold-Moyal extension respects the algebraic constraints, but also show that there is an additional term passing the algebraic filter and this term can in no way be generated by the Groenewold-Moyal product.

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2 The classical action and the symmetries

In the BRS approach in the Landau gauge [6], the classical action of the BF theory over a two dimensional Euclidean spacetime is:

$$S_{BF} = \frac{1}{2} \int d^2x \; \epsilon^{\mu\nu} F^a_{\mu\nu} \; \phi_a + \int d^2x \; s \; (\bar{c}_a \; \partial^\mu A^a_\mu)$$
$$+ \int d^2x \; \left[\Omega^\mu_a (sA^a_\mu) + L^a(sc_a) + \rho^a(s\phi_a) \right]$$

(1)

where $\epsilon^{\mu\nu}$ is the completely antisymmetric Levi-Civita tensor ($\epsilon^{12} = +1$). All fields belong to the adjoint representation of the gauge group $\mathcal{G}$. In particular ϕ^a are scalar fields and $F^a_{\mu\nu}$ is the field strength

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu$$

(2)

with f^{abc} completely antisymmetric real structure constants of $\mathcal{G}$. The product rule for the generators τ^a of the Lie algebra of $\mathcal{G}$ is

$$\tau^a \tau^b = \frac{i}{2} \; f^{abc} \tau_c + \frac{1}{2} \; d^{abc} \tau_c \qquad a, b, c = 1, \ldots, dim(\mathcal{G})$$

(3)

so that

$$[\tau^a, \tau^b] = i f^{abc} \tau_c \quad \{\tau^a, \tau^b\} = d^{abc} \tau_c$$

(4)

with d^{abc} a completely symmetric tensor of rank 3, and $Tr(\tau^a \tau^b) = \delta^{ab}$. The ghost fields c^a and the antighost fields $\bar{c}^a$ have a Faddeev-Popov charge respectively equal to $+1$ and -1. The external fields Ω^a_μ, L^a e ρ^a are introduced to take into account the nonlinearity of the BRS transformations:

$$sA^a_\mu = -(D_\mu c)^a = -(\partial_\mu c^a + f^{abc} A^b_\mu c^c)$$
$$s\phi^a = f^{abc} c^b \phi^c$$
$$sc^a = \frac{1}{2} \; f^{abc} c^b c^c$$
$$s\bar{c}^a = b^a$$
$$sb^a = 0$$

(5)

The fields b^a are the Lagrange multipliers for the gauge condition. With the above definition the s operator is nilpotent [7]

$$s^2 = 0$$

(6)

The action (1) is characterized by the following set of symmetries and constraints: [6]

1. BRS invariance, expressed by the Slavnov-Taylor identity

$$\mathcal{S}(S_{BF}) =$$
$$= \int d^2x \left(\frac{\delta S_{BF}}{\delta \Omega_\mu^a} \frac{\delta S_{BF}}{\delta A_\mu^a} + \frac{\delta S_{BF}}{\delta \rho^a} \frac{\delta S_{BF}}{\delta \phi^a} + \frac{\delta S_{BF}}{\delta L^a} \frac{\delta S_{BF}}{\delta c^a} + b^a \frac{\delta S_{BF}}{\delta \bar{c}^a} \right) = 0 \tag{7}$$

2. the Landau gauge

$$\frac{\delta S_{BF}}{\delta b^a(x)} = \partial^\mu A_\mu^a(x) \tag{8}$$

3. the ghost equation of motion [8], which holds true in the Landau gauge

$$\int d^2x \left(\frac{\delta}{\delta c^a} + f^{abc} \, \bar{c}^b \frac{\delta}{\delta b^c} \right) S_{BF} \equiv \mathcal{G}^a S_{BF} = \Delta^a \tag{9}$$

where

$$\Delta^a = \int d^2x \, f^{abc} \, (\Omega_\mu^b A_\mu^c - L^b c^c - \rho^b \phi^c) \tag{10}$$

4. the antighost equation of motion

$$\left(\frac{\delta}{\delta \bar{c}^a(x)} + \partial^\mu \frac{\delta}{\delta \Omega_\mu^a(x)} \right) S_{BF} \equiv \bar{\mathcal{G}}^a(x) S_{BF} = 0 \tag{11}$$

This condition is not independent from the others: it can be derived from the commutator between the Slavnov-Taylor operator and the gauge condition [7]

5. the supersymmetry

$$\mathcal{W}_\mu S_{BF} = \Delta_\mu \tag{12}$$

where

$$\mathcal{W}_\mu = \int d^2x \left(\epsilon_{\mu\nu} \, \rho^a \frac{\delta}{\delta A_\nu^a} - \epsilon^{\mu\nu} \, (\Omega_\nu^a + \partial_\nu \bar{c}^a) \frac{\delta}{\delta \phi^a} \right.$$
$$\left. - A_\mu^a \frac{\delta}{\delta c^a} + (\partial_\mu \bar{c}^a) \frac{\delta}{\delta b^a} - L^a \frac{\delta}{\delta \Omega_\mu^a} \right) \tag{13}$$

$$\Delta_\mu = \int d^2x \, [L^a(\partial_\mu c^a) - \rho^a \partial_\mu \phi^a - \Omega_\nu^a(\partial_\mu A_\nu^a) - \epsilon_{\mu\nu} \rho^a \partial^\nu b^a] \tag{14}$$

the existence of which is due to the topological nature of the BF model and to the choice of the Landau gauge [9]

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We also note that the breakings Δ^a and Δ_μ, being linear in the quantum fields, will be present only at the classical level [7] .

The whole set of all these symmetries can be summarized in a closed algebra with breakings, that, for a generic even Faddeev-Popov charged functional γ, can be expressed as:

$$
\begin{aligned}
& B_\gamma \mathcal{S}(\gamma) = 0 \\
& \mathcal{W}_\mu \mathcal{S}(\gamma) + B_\gamma (\mathcal{W}_\mu \gamma - \Delta_\mu) = P_\mu \gamma \\
& \mathcal{G}^a \mathcal{S}(\gamma) + B_\gamma (\mathcal{G}^a \gamma - \Delta^a) = \mathcal{H}^a \gamma \\
& \mathcal{G}^a (\mathcal{W}_\mu \gamma - \Delta_\mu) + \mathcal{W}_\mu (\mathcal{G}^a \gamma - \Delta^a) = 0 \\
& B_\gamma \left(\frac{\delta \gamma}{\delta b^a} - \partial^\mu A_\mu^a \right) - \frac{\delta}{\delta b^a} \, \mathcal{S}(\gamma) = \bar{\mathcal{G}}^a \gamma \\
& \{ \mathcal{G}^a, \mathcal{G}^b \} = 0 \\
& \{ \mathcal{W}_\mu, \mathcal{W}_\nu \} = 0 \\
& [\mathcal{H}^a, \mathcal{G}^b] = -f^{abc} \mathcal{G}_c \\
& [\mathcal{H}^a, \mathcal{H}^b] = -f^{abc} \mathcal{H}_c
\end{aligned}
\tag{15}
$$

where B_γ is the linearized Slavnov-Taylor operator

$$
\begin{aligned}
B_\gamma = \int d^2 x \; & \left[\frac{\delta \gamma}{\delta \Omega_\mu^a} \frac{\delta}{\delta A_\mu^a} + \frac{\delta \gamma}{\delta A_\mu^a} \frac{\delta}{\delta \Omega_\mu^a} + \frac{\delta \gamma}{\delta \rho^a} \frac{\delta}{\delta \phi^a} + \frac{\delta \gamma}{\delta \phi^a} \frac{\delta}{\delta \rho^a} \right. \\
& \left. + \frac{\delta \gamma}{\delta L^a} \frac{\delta}{\delta c^a} + \frac{\delta \gamma}{\delta c^a} \frac{\delta}{\delta L^a} + b^a \frac{\delta}{\delta \bar{c}^a} \right]
\end{aligned}
\tag{16}
$$

the operator $\mathcal{H}^a$ expresses a global gauge transformation

$$
\mathcal{H}^a = \sum_{(all\ fields\ \psi)} \int d^2 x \; f^{abc} \, \psi^b \frac{\delta}{\delta \psi^c}
\tag{17}
$$

and finally P_μ is a global translation

$$
P_\mu = \sum_{(all\ fields\ \psi)} \int d^2 x \; (\partial_\mu \psi^a) \frac{\delta}{\delta \psi_a}
\tag{18}
$$

As shown in [6], the symmetries in (15) allow a full quantum extension of the theory. In fact it can be proved that the action in (1) is stable and that the symmetries are not anomalous. [1]

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[1] In effect this theory can be proved to be perturbatively finite

3 Noncommutative extension by means of the Groenewold-Moyal product

In order to extend the classical action (1) to a noncommutative spacetime we have to define a new prescription for the product of fields. A popular choice in the literature is the Groenewold-Moyal star product defined by [2, 3]

$$
\begin{aligned}
f(x) \star g(x) &= f(x) \, e^{\frac{i}{2} \overleftarrow{\partial_i} \theta^{ij} \overrightarrow{\partial_j}} \, g(x) \\
&= f(x)g(x) + \frac{i}{2} \, \theta^{ij} \partial_i f(x) \partial_j g(x) + O(\theta^2)
\end{aligned}
\tag{19}
$$

where θ^{ij} is an antisymmetric real constant tensor, with the dimensions of an inverse squared mass, defined by

$$
[\hat{x}^i, \hat{x}^j] = i \, \theta^{ij}
\tag{20}
$$

At the leading order in θ, we find that the noncommutative extension of the classical action (1) according to the Groenewold-Moyal star product is

$$
\widehat{S}_{BF} = S_{BF} + \theta^{\rho\sigma} X_{\rho\sigma}^{(GM)} + O(\theta^2)
\tag{21}
$$

where

$$
\begin{aligned}
X_{\rho\sigma}^{(GM)} &= \frac{1}{4} \int d^2x \, d^{abc} \epsilon^{\mu\nu} (\partial_\sigma A_\nu^a)(\partial_\rho A_\mu^b) \, \phi^c \\
&+ \frac{1}{2} \int d^2x \, d^{abc} \, (\partial_\mu \bar{c}^a)(\partial_\rho c^b)(\partial_\sigma A_\mu^c) + \frac{1}{2} \int d^2x \, d^{abc} \, \Omega_\mu^a (\partial_\rho c^b)(\partial_\sigma A_\mu^c) \\
&+ \frac{1}{4} \int d^2x \, d^{abc} \, L^a (\partial_\rho c^b)(\partial_\sigma c^c) + \frac{1}{2} \int d^2x \, d^{abc} \, \rho^a (\partial_\rho c^b)(\partial_\sigma \phi^c)
\end{aligned}
\tag{22}
$$

4 Symmetries of the noncommutative action

Generally speaking, we may say that the noncommutative extension of our classical action S_{BF} has the following form:

$$
\widehat{S}_{BF} = S_{BF} + \theta^{\rho\sigma} X_{\rho\sigma} + O(\theta^2)
\tag{23}
$$

where $X_{\rho\sigma}$ is a local functional of the fields, with canonical dimension less than or equal to four ($[\theta^{\rho\sigma}] = -2$) and Faddeev-Popov charge zero. In the previous section we have found a particular extension $X_{\rho\sigma}^{(GM)}$, which derives from the use of the Groenewold-Moyal product. This approach is neither the most general, nor the unique one. In this paper we want to present an alternative way of constructing the correction $X_{\rho\sigma}$ without introducing any particular star product a priori defined. Our point of view, which is

borrowed from the standard approach to the commutative theory, is that the symmetries themselves, once correctly defined to take into account the $\theta^{\rho\sigma}$ tensor, should be used as guidelines to characterize the $X_{\rho\sigma}$ term. The main ingredient is thus the algebra in (15), which we want to preserve. Of course the definition of the symmetries in (15) when $\theta^{\rho\sigma}$ is present has a certain degree of arbitrariness. Our choice is "minimal", in the sense that we decide to keep unchanged the functional form of the operators that express the symmetries and constraints on S_{BF}, and the introduction in (23) of the tensor $\theta^{\rho\sigma}$ can only modify the classical breakings

$$\frac{\delta\widehat{S}_{BF}}{\delta b^a(x)} = \partial^\mu A_\mu^a(x) + \theta^{\mu\nu}\Xi_{\mu\nu}^a(x) + O(\theta^2)$$

$$\mathcal{G}^a\widehat{S}_{BF} = \Delta^a + \theta^{\mu\nu}\int d^2x\ \Delta_{\mu\nu}^a(x) + O(\theta^2) \tag{24}$$

$$\mathcal{W}_\mu\widehat{S}_{BF} = \Delta_\mu + \theta^{\rho\sigma}\int d^2x\ (\Lambda_\mu)_{\rho\sigma}(x) + \theta^{\mu\tau}\int d^2x\ \Lambda_\tau(x) + O(\theta^2)$$

A detailed analysis of the first order terms in $\theta^{\mu\nu}$ in (24) shows that none of them is allowed, if we wish to preserve the algebraic structure in (15). This can be proved by explicitly evaluating the commutators and the anticommutators in (15) with the objects defined in (24) and enforcing the validity of (15). We may now assume that all the symmetries characterizing S_{BF} can be extended to $\widehat{S}_{BF}$. This is of great relevance because it expresses a strong constraint on the choice of the possible corrections $X_{\rho\sigma}$ to the ordinary action.

5 Comparison with the Groenewold-Moyal extension

A direct calculation shows that the noncommutative corrections derived with the use of the Groenewold-Moyal star product are all compatible with the symmetries characterizing $\widehat{S}_{BF}$. However they are not the only ones. For example a term of the form

$$\theta^{\rho\sigma}\int d^2x\ F_{\rho\mu}^a\epsilon^{\mu\nu}F_{\nu\sigma}^a \tag{25}$$

is compatible with the constraints, but there is no way to derive it from the Groenewold-Moyal product. The presence of a term of the form (25), that is compatible with all of the algebraic constraints, implies that the noncommutative extensions of the BF model based on the Groenewold-Moyal star product are not stable. Consequently their quantum extensions cannot be correctly defined since the coupling (25), needed as a counterterm, is not present at the classical level [7]. On the other hand, if we add (25) to

the classical action we definitely spoil the Groenewold-Moyal star product extension.

6 Conclusions

In this letter we have shown that the noncommutative extension of the two dimensional BF model based on the Groenewold-Moyal product is not stable in the sense that, already at the first order in $\theta^{\mu\nu}$, the resulting classical action is not the most general local functional respecting power counting and satisfying the algebraic constraints which define the model. Two final remarks are in order: first the term in (25) can be rewritten (due to the two spacetime dimensions) as

$$\theta^{12} \int d^2x \ F^a_{\mu\nu} \ F^{\mu\nu}_a \tag{26}$$

which shows that the noncommutative stable extension acquires a contribution which in two spacetime dimensions has still a topological character, but in a more general context is not topological, suggesting that some of the symmetries could be broken at the quantum level. Second, the same term in (25) will propagate to all orders in $\theta^{\mu\nu}$, by simply taking powers of $\theta^{\mu\nu}X_{\mu\nu}$ and mixes with the Groenewold-Moyal contributions. This suggests that, at an arbitrary order in $\theta^{\mu\nu}$, we will find non Groenewold-Moyal terms whose coefficient is free and thus an infinite number of couplings. In our opinion this opens a serious problem concerning the noncommutative extensions of quantum field theory models. Work is in progress to completely characterize the BF model. The analysis will include not only stability but also the anomaly issue. In particular the possible presence of anomalous terms will definitely spoil the renormalizability of the noncommutative extension, contrary to the standard commutative case.

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